

①

$$H(V) = I - 2VV^*, \quad V^*V = I$$

$$\begin{aligned} H(V)H^*(V) &= (I - 2VV^*)(I - 2(VV^*)^*) = \\ &= (I - 2VV^*)(I - 2VV^*) = I - 4VV^* + 4VV^*VV^* = \\ &= I - 4VV^* + 4VV^* = I \end{aligned}$$

$$(VV^*)_{ij}^* = \overline{(VV^*)_{ji}} = \overline{V_j V_i^*} = V_j^* V_i = V_i V_j^* = (VV^*)_{ij}$$

\Rightarrow unitary \Rightarrow full rank

②

$$a) \|x\|_2 = \sqrt{\sum_{i=1}^m |x_i|^2}; \quad \|x\|_\infty = \max_{i=1, \dots, m} |x_i|$$

$$\sum_{i=1}^m |x_i|^2 \leq m \cdot \max_{i=1, \dots, m} |x_i|^2 \Rightarrow \|x\|_2 \leq \sqrt{m} \|x\|_\infty$$

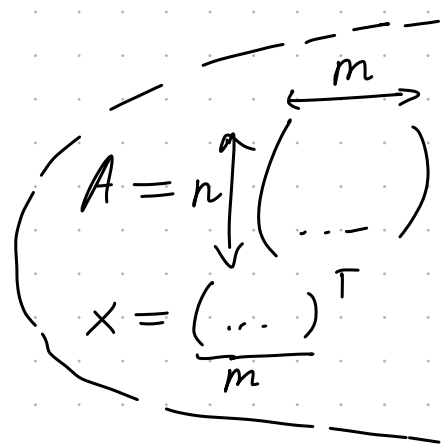
$$x = (\underbrace{1, 1, \dots, 1}_m)^T \Rightarrow \|x\|_2 = \sqrt{m}; \quad \|x\|_\infty = 1$$

$$b) \|A\|_\infty = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \sup_{x \neq 0} \frac{\max_{i=1, \dots, m} |(Ax)_i|}{\max_{i=1, \dots, m} |x_i|}$$

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

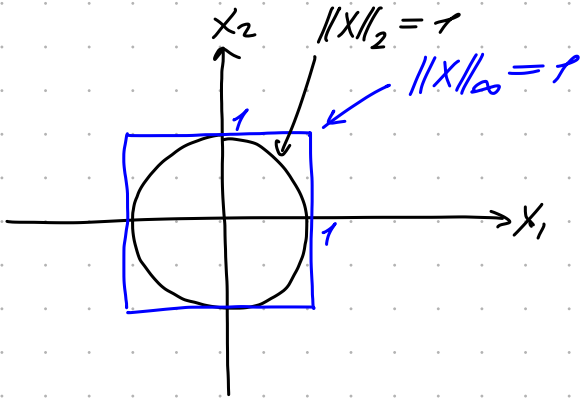
$$\|\alpha x\|_* = |\alpha| \|x\|_* \Rightarrow \|Ax\|_* = \|x\|_* \|Au\|_*, \quad \text{where } u = \frac{x}{\|x\|_*}$$

$$\Rightarrow \|A\|_* = \sup_{\|x\|_* = 1} \|Ax\|_* - \text{can calculate norm on unit ball}$$



$$\|Ax\|_2 = \sup_{\|x\|_2=1} \sqrt{\sum_{i=1}^n \left| \sum_{j=1}^m A_{ij} x_j \right|^2} \leq \sup_{\|x\|_2=1} \sqrt{n \max_{i=1, \dots, n} \left| \sum_{j=1}^m A_{ij} x_j \right|^2} \leq$$

$$\leq \sup_{\|x\|_\infty=1} \sqrt{n \max_{j=1}^m \left| \sum_{i=1}^n A_{ij} x_j \right|^2} = \sqrt{n} \|Ax\|_\infty$$



$$A = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}_{m \times m} \quad (Ax)_i = \sum_{j=1}^m x_j$$

$$\Rightarrow \|Ax\|_\infty = \sup_{\|x\|_\infty=1} \sum_{j=1}^m |x_j| = n\sqrt{m}$$

$$\|Ax\|_2 = \sup_{\|x\|_2=1} \sqrt{\sum_{i=1}^n \left| \sum_{j=1}^m x_j \right|^2} = \sqrt{n} \sup_{\|x\|_2=1} \left| \sum_{j=1}^m x_j \right|$$

$$= \sqrt{n} \sup_{\|x\|_2=1} \left| \sum x_j \right|$$

$$\frac{\sum |x_j|}{m} \leq \sqrt{\frac{\sum |x_j|^2}{m}} = \frac{1}{\sqrt{m}} \quad \text{— mean inequality}$$

for $\|x\|_2=1$

$$\Rightarrow \sqrt{n} \sup_{\|x\|_2} \left| \sum x_j \right| = \sqrt{n} \sup_{\|x\|_2=1} \sum |x_j| = \sqrt{n} \sqrt{m}$$

$$\Rightarrow \|A\|_\infty = n\sqrt{m} = \sqrt{n} \|A\|_2$$

③

$$A = I + uv^*$$

$$u = (-1, 0, \dots, 0), \quad v = (1, \dots, 0) \Rightarrow uv^* = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & & 0 \end{pmatrix}$$

$$\Rightarrow I + uv^* = \text{diag}(0, 1, \dots, 1) \Rightarrow \text{can be singular}$$

$$(I + uv^*)(I + \lambda uv^*) = I + (\lambda + 1 + \lambda v^*u)uv^* = I \Rightarrow$$

$$\Rightarrow \lambda = -\frac{1}{1 + v^*u}$$

if $v^*u = -1 \Rightarrow$ „vse ploho“ (*)

$$\text{else} \Rightarrow A^{-1} = 1 - \frac{uv^*}{1+v^*u} \quad \left(1 + uv^* = 1 + u_1\bar{v}_1 + u_2\bar{v}_2 + u_3\bar{v}_3 \right)$$

$$\det A = \det (1 + uv^*) \stackrel{?}{=} (1 + v^*u)$$

$$\det (1 + uv^*) = \begin{vmatrix} 1 + u_1\bar{v}_1 & u_1\bar{v}_2 & u_1\bar{v}_3 \\ u_2\bar{v}_1 & 1 + u_2\bar{v}_2 & u_2\bar{v}_3 \\ u_3\bar{v}_1 & u_3\bar{v}_2 & 1 + u_3\bar{v}_3 \end{vmatrix} =$$

$$= \left(\underline{1 + u_2\bar{v}_2 + u_3\bar{v}_3} - \cancel{u_3\bar{v}_2 u_2\bar{v}_3} + \cancel{u_1\bar{v}_1} + \cancel{u_1\bar{v}_1 u_2\bar{v}_2 u_3\bar{v}_3} \right.$$

$$\left. + \cancel{u_1\bar{v}_1 u_2\bar{v}_2} + \cancel{u_1\bar{v}_1 u_3\bar{v}_3} - \cancel{u_1\bar{v}_1 u_3\bar{v}_2 u_2\bar{v}_3} \right) -$$

$$- (u_1\bar{v}_2) \left(\cancel{u_2\bar{v}_1} + \cancel{u_2\bar{v}_1 u_3\bar{v}_3} - \cancel{u_2\bar{v}_3 u_3\bar{v}_1} \right) +$$

$$+ u_1\bar{v}_3 \left(\cancel{u_2\bar{v}_1 u_3\bar{v}_2} - \cancel{u_3\bar{v}_1} - \cancel{u_2\bar{v}_2 u_3\bar{v}_1} \right) =$$

$$= 1 + v^*u \Rightarrow \det (1 + uv^*) \neq 0 \Leftrightarrow 1 + v^*u \neq 0 \quad (*)$$

④

$$\|UA\|_F = \|B\|_F = \left\| \begin{pmatrix} b_1 & \dots & b_n \end{pmatrix} \right\|_F = \sqrt{\sum_{i,j} |b_{ij}|^2} = \sqrt{\sum_{i=1}^n \|b_i\|_2^2}$$

$$b_{ij} = \sum_k u_{ik} A_{kj} \Rightarrow b_i = \begin{pmatrix} \sum_k u_{ik} A_{k1} \\ \dots \\ \sum_k u_{ik} A_{kn} \end{pmatrix} = UA_i$$

$$\|b_i\|_2 = \|UA_i\|_2 =$$

$$= \sup_{\|x\|_2=1} \|UA_i x\|_2, \quad \|UA_i x\|_2^2 = (UA_i x)^* (UA_i x) = \bar{x} A_i^+ U^+ UA_i x$$

$$= (\bar{x} A^+ A x) = \|Ax\|_2^2$$

$$\Rightarrow \|b_i\|_2 = \|A_i\|_2 \Rightarrow \|UA\|_F = \|A\|_F$$

$$\|AU\|_F = \|U^T AU\|_F$$

$$U^T AU = U^{-T} AU \Rightarrow \|U^{-T} AU\|_F = \|A'\|_F = \sup_{\|x'\|_2=1} \|A'x'\|_2 =$$

$$= \sup_{\|x\|_2=1} \|Ax\|_2 = \|A\|_F$$

unitary transform $\Rightarrow \|Ax\|_2 = \|A'x'\|_2$
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