HW #1 №1,2,3,4

①

$$H(v) = 1 - 2vv^*, \quad v^*v = 1 \qquad (vv^*)^* =$$

$$H(v)H^*(v) = (1 - 2vv^*)(1 - 2(vv^*)^*) =$$

$$= (1 - 2vv^*)(1 - 2vv^*) = 1 - 4vv^* + 4vv^*vv^* =$$

$$= 1 - 4vv^* + 4vv^* = 1$$

$$(vv^*)^*_{ij} = \overline{(vv^*)_{ji}} = \overline{v_j v_i^*} = v_j^* v_i = v_i v_j^* = (vv^*)_{ij}$$

$\Rightarrow$ unitary $\Rightarrow$ full rank

②

a) $\|x\|_2 = \sqrt{\sum_{i=1}^{m} |x_i|^2}$; $\quad \|x\|_\infty = \max_{i=\overline{1,m}} |x_i|$

$$\sum_{i=1}^{m} |x_i|^2 \le m \cdot \max_{i=\overline{1,m}} |x_i|^2 \Rightarrow \|x\|_2 \le \sqrt{m}\,\|x\|_\infty$$

$$x = \underbrace{(1, 1, \dots, 1)}_{m}^T \Rightarrow \|x\|_2 = \sqrt{m}; \ \|x\|_\infty = 1$$

б) $\|A\|_\infty = \sup_{x \ne 0} \dfrac{\|Ax\|_\infty}{\|x\|_\infty} = \sup_{x \ne 0} \dfrac{\max_{i=\overline{1,m}} |(Ax)_i|}{\max_{i=\overline{1,m}} |x_i|}$

$$A = n\updownarrow \overset{\xleftrightarrow{m}}{\left(\ \dots \ \right)}$$

$$x = \underbrace{(\ \dots\ )}_{m}^T$$

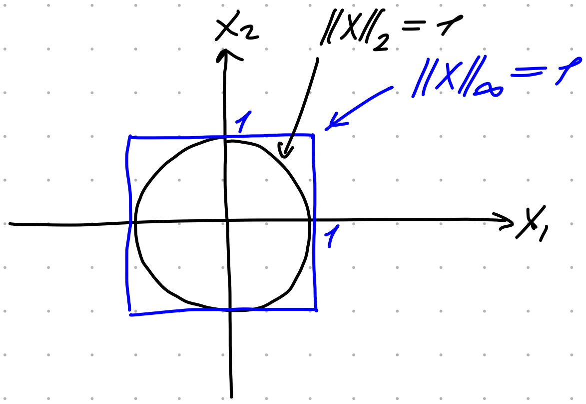
$$\|A\|_2 = \sup_{x \ne 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$\|\alpha x\|_* = |\alpha|\,\|x\| \Rightarrow \|Ax\|_* = \|x\|_* \|Au\|_*$, where $u = \dfrac{x}{\|x\|_*}$

$\Rightarrow \|A\|_* = \sup_{\|x\|_* = 1} \|Ax\|_*$ — can calculate norm on unit ball

$$\|Ax\|_2 = \sup_{\|x\|_2=1} \sqrt{\sum_{i=1}^{n} \Big|\sum_{j=1}^{m} A_{ij} x_j\Big|^2} \le \sup_{\|x\|_2=1} \sqrt{n \max_{i=\overline{1,n}} \Big|\sum_{j=1}^{m} A_{ij} x_j\Big|^2} \le$$

$$\le \sup_{\|x\|_\infty=1} \sqrt{n \max \Big|\sum_{j=1}^{m} A_{ij} x_j\Big|^2} = \sqrt{n}\, \|Ax\|_\infty$$



$$A = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{m \times m} \qquad (Ax)_i = \sum_{j=1}^{m} x_j$$

$$\Rightarrow \|Ax\|_\infty = \sup_{\|x\|_\infty=1} \sum_{i=1}^{n} \Big|\sum_{j=1}^{m} x_j\Big| = n\sqrt{m}$$

$$\|Ax\|_2 = \sup_{\|x\|_2=1} \sqrt{\sum \Big|\sum_{j=1}^{m} x_j\Big|^2} = \sqrt{n} \sup_{\|x\|_2} \sqrt{\Big|\sum_{j=1}^{m} x_j\Big|^2}$$

$$= \sqrt{n} \sup_{\|x\|_2=1} \Big|\sum x_j\Big|$$

$$\frac{\sum |x_j|}{m} \le \sqrt{\frac{\sum |x_j|^2}{m}} = \frac{1}{\sqrt{m}} \quad \text{— mean inequality}$$

for $\|x\|_2 = 1$

$$\Rightarrow \sqrt{n} \sup_{\|x\|_2} \Big|\sum x_j\Big| = \sqrt{n} \sup_{\|x\|_2=1} \sum |x_j| = \sqrt{n}\sqrt{m}$$

$$\Rightarrow \|A\|_\infty = n\sqrt{m} = \sqrt{n}\, \|A\|_2$$

3

$$A = 1 + uv^*$$

$$u = (-1, 0, \ldots, 0),\ v = (1, \ldots 0) \Rightarrow uv^* = \begin{pmatrix} -1 & 0 \ldots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix}$$

$\Rightarrow 1 + uv^* = \mathrm{diag}(0, 1, \ldots, 1) \Rightarrow$ can be singular

$$(1 + uv^*)(1 + \alpha uv^*) = 1 + (\alpha + 1 + \alpha v^* u) uv^* = 1 \Rightarrow$$

$$\Rightarrow \alpha = -\frac{1}{1 + v^* u}$$

if $v^* u = -1 :\Rightarrow$ „vse ploho" $(*)$

$$\text{else} \Rightarrow A^{-1} = 1 - \frac{uv^*}{1+v^*u} \qquad \left(1 + uv^* = 1 + u_1\bar{v}_1 + u_2\bar{v}_2 + u_3\bar{v}_3\right)$$

$$\det A = \det(1+uv^*) \overset{?}{\propto} (1+v^*u)$$

$$\det(1+uv^*) = \begin{vmatrix} 1+u_1\bar{v}_1 & u_1\bar{v}_2 & u_1\bar{v}_3 \\ u_2\bar{v}_1 & 1+u_2\bar{v}_2 & u_2\bar{v}_3 \\ u_3\bar{v}_1 & u_3\bar{v}_2 & 1+u_3\bar{v}_3 \end{vmatrix} =$$

$$= \Big(1 + \cancel{u_2\bar{v}_2 u_3\bar{v}_3} + u_2\bar{v}_2 + u_3\bar{v}_3 - \cancel{u_3\bar{v}_2 u_2\bar{v}_3} + u_1\bar{v}_1 + \cancel{u_1\bar{v}_1 u_2\bar{v}_2 u_3\bar{v}_3}$$
$$+ \cancel{u_1\bar{v}_1 u_2\bar{v}_2} + \cancel{u_1\bar{v}_1 u_3\bar{v}_3} - \cancel{u_1\bar{v}_1 u_3\bar{v}_2 u_2\bar{v}_3}\Big) -$$

$$- (u_1\bar{v}_2)\left(\cancel{u_2\bar{v}_1} + \cancel{u_2\bar{v}_1 u_3\bar{v}_3} - \cancel{u_2\bar{v}_3 u_3\bar{v}_1}\right) +$$

$$+ u_1\bar{v}_3\left(\cancel{u_2\bar{v}_1 u_3\bar{v}_2} - \cancel{u_3\bar{v}_1} - \cancel{u_2\bar{v}_2 u_3\bar{v}_1}\right) =$$

$$= 1 + v^*u \Rightarrow \det(1+uv^*) \neq 0 \Leftrightarrow 1+v^*u \neq 0 \quad (*)$$

4

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} =$$

$$(AA^*)_{jj} = \sum a_{ij} a_{ij}^* = \sum_i |a_{ij}|^2$$

$$\Rightarrow \|A\|_F = \sqrt{\operatorname{trace}(AA^*)} = \sqrt{\operatorname{trace}(A^*A)}$$

$$\sqrt{\operatorname{trace}(AUU^*A^*)} \qquad\qquad \sqrt{\operatorname{trace}(A^*U^*UA)}$$

$$\sqrt{\operatorname{trace}[AU(AU)^*]} = \|AU\|_F \qquad\qquad \sqrt{\operatorname{trace}[(UA)^*UA]} = \|UA\|_F$$

$UU^* = U^*U = 1$ - unitary matrix