Numerical Methods, Fall 2022
Assignment 2 [SVD decomposition with applications]
Total: 40, Deadline: 21 Oct

SUGGESTED READING

- Lectures 4-5 of [1]
- Lecture 2 of [2]
- Making sense of principal component analysis, eigenvectors and eigenvalues

EXERCISES

1. (10) In this exercise, we will explore three main algorithms available in scientific python distributions for computation of SVD: numpy.linalg.svd, scipy.sparse.linalg.svds and sklearn.utils.extmath.randomized_svd. To this end:
   - Construct a random $n \times n$ matrix A (with iid elements sampled from standard normal distribution); consider $n = 2000$.
   - Using these implementations, construct rank-2 approximations to A. You will thus obtain three rank-2 matrices A_{svd}, A_{svds} and A_{rsvd}. Measure the run–time of these three algorithms for the given task.
   - Compute the error norms: $\|A - A_{\text{svd}}\|_F$, $\|A - A_{\text{svds}}\|_F$, $\|A - A_{\text{rsvd}}\|_F$. Explain the result.
2. (5) Let A be $m \times n$ with SVD $A = U\Sigma V^T$. Compute SVDs of the following matrices in terms of U, Σ and V: (i) $\left(A^T A\right)^{-1}$, (ii) $\left(A^T A\right)^{-1} A^T$, (iii) $A\left(A^T A\right)^{-1}$, (iv) $A\left(A^T A\right)^{-1} A^T$.
3. (10) Consider the matrix:

$$\begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$$

   - List the singular values, left singular vectors and right singular vectors of A. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V.
   - Draw a labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors with the coordinates of their vertices marked.
   - What are 2-norm and Frobenius norm of A?
   - Find A^{-1} not directly, but via SVD.
   - Find the eigenvalues λ_1, λ_2 of A.
4. (5) The file `A.npy` contains the $n \times n$ matrix A. Determine the best approximation of A_{ij} in terms of the following anzats, where the variables are separated: $A_{ij} \approx h_i \eta_j$. What is the related relative error of such approximation:

$$\delta_{\text{err}} = \frac{\sqrt{\sum_{ij} \left(A_{ij} - h_i \eta_j\right)^2}}{\sqrt{\sum_{ij} A_{ij}^2}}?$$

   How many terms K would an exact representation of the following form:

$$A_{ij} = \sum_{\alpha=1}^{K} h_{\alpha i} \eta_{\alpha j}$$

   require?

5. (10) In this exercise, you will explore application of SVD to dimensionality reduction. Let us start with loading the dataset:

```
from sklearn.datasets import load_digits
digits = load_digits()
A = digits.data
y = digits.target
```

so that rows of A contain monochromatic images of digits (64 float values which should be reshaped into 8×8 images) and y contains the digit labels.

- Inspect the dataset: plot examples of images, corresponding to several digits (say $0, 3, 7$).
- Normalize the dataset A.
- Use SVD to project the dataset A from 64 dimensions to 2 dimensions. Show the colored scatter plot, where colors encode the digits.

REFERENCES

[1] Lloyd N Trefethen and David Bau III. *Numerical linear algebra*. Vol. 50. Siam, 1997.

[2] Eugene E Tyrtyshnikov. *A brief introduction to numerical analysis*. Springer Science & Business Media, 2012.